Selling separately as a robust mechanism for a multi-product monopoly

Yang Yu*

Abstract

This paper proposes selling separately as a robust mechanism for a multi-product seller who faces uncertainty of the correlations between product values. In the model, a deterministic mechanism, i.e. price schedule, is offered by a seller to a buyer which has private information about product values. We prove that in the two-item case with continuous consumer types, to maximize the worst-case expected profit, the best strategy is to sell independently if the seller only knows the marginal distribution of each item's valuation. The proof consist of two parts. The first part proves that for any mechanism that demands bundle premium, there exists a joint distribution, especially independent distribution applying for all mechanisms, that generates expected profit no more than that of optimally selling each item separately. This shows that if the valuations are independently distributed and known by the seller, offering bundling premium is always dominated by selling separately. The second part proves that for any mechanism that offers bundle discounts, there also exists a joint distribution that generates expected profit no more than that of optimally selling each item separately.

Keywords: Bundling, Robust Mechanism, Price Schedule

JEL classification:

*Stony Brook University. Email: yang.yu.2@stonybrook.edu.

1 Introduction

To sell goods by bundling them together is a common feature in many product and service markets. For example, softwares, like Mircosoft words, excels, or other presentation programs are always bundled into a single office suite. New cars are sold with various option packages that bundle options such as leather seats, DVD/Navigation Systems, and hi-end sound systems. McDonald's or KFC combine different items into 'meals'. A seller can offer a bundle of goods with a price that is higher (bundle premium) or lower (bundle discount) than the sum of prices for each individual good. Whether and how to offer bundle premium, or bundle discount or to sell individual good independently is an important decision by a multiproduct monopoly.

Typical rationales for bundling behaviors include synergy on both seller side and buyer side. On the buyer side, if the goods are complementary then it's more efficient to bundle goods together. On the seller side, there may exist the economies of scope in producing or distributing the goods together. If synergy effect is ignored, a reliable explanation for bundling is that it can serve as a price discrimination device to screen buyers. Adams and Yellen (1976) first analyzed the effect of bundling by many examples. They show that commodity bundling can impove seller's revenue especially when the bundled goods have a negative correlation in value. After that, McAfee et al. (1989) investigated the conditions under which bundling is an optimal selling strategy. Their results show that if the valuations are independent, then bundling will strictly dominate selling separately.

This paper also studies the bundling behavior from the perspective of price discrimination. We consider a situation in which the multiproduct seller faces uncertainty of the correlations between product values. The potential applications for this setup include the market in which buyers' willingness to pay for each product are all affected by some common unknown factors such as income or when a firm enters the market of a new product that usually sold bundled with another product.

In the model, a deterministic mechanism, i.e. pricing schedule, is offered by a seller to a buyer which has private information about product values. The seller knows the marginal distribution of the individual items but does not know the correlations between (the joint distribution of) the valuations of the two items. In this situation, we consider seller as a pessimist who maximizes the worst-case expected profit, which means that he will choose a mechanism that maximizes expected profit given that the joint distribution he conjectures will generate a minimum profit under the mechanism he chooses over the entire class of mechanisms. Our result shows that, to maximize the worst-case expected profit, the best strategy is to sell independently if the seller only knows the marginal distribution of the item valuation. If the valuations are independently distributed and known by the seller, offering bundling premium is dominated by selling separately.

As mentioned before, this paper contributes to a tranditional literature of bundling. After Adams and Yellen (1976) and McAfee et al. (1989), some papers examined the use of bundling as a way to induce self-selection among heterogenous types of consumers. Kolay and Shaffer (2003) showed that compared to a strategy that uses menus of two-part tariffs, bundling yields higher profits for the monopolist. Following Manelli et al. (2006), we consider a determinisitic mechanism called price schedule which is a collection of prices, one price per bundle. The buyer can choose the option he likes or quit(the outside option) but he can only choose one out of these options. By this definition, we avoid the discussion about the concepts of mixed bundling or pure bundling.

This paper also contributes to a growing literature of robust mechanism design. Bergemann and Morris (2005) first points out that the tranditional mechanism design literature assumes too much common knowledge of the environment among the players and the planner. They examine the implications of relaxing this strong informational assumption and the applications in the auction design. This paper applies the philosophy to the pricing problem of a multi-product monopoly. Assuming that the monopoly only has partial information about buyers' valuations (knows the marginal distribution but doesn't know the joint distribution), we explore the seller's 'optimal' pricing strategy that achieves the highest performance guarantee which is robust to any informational assumptions.

The structure of paper is the following: Section 2 introduces the basic model; Section 3 provides an example with discrete type buyers to illustrate the idea; Section 4 introduces the general model with continous types and proposes selling separately as a robust mechanism for a pessimistic seller; Section 5 provides a graphical proof; Section 6 concludes.

2 Preliminary

A monopoly sells two indivisible items, item 1 and item 2, to a single buyer. The buyer's valuation or reservation price for the two items are repectively v_1, v_2 and are the buyer's private information. The buyer has unit demand for each item and additive valuation[1] for the two items.

The seller has zero valuation for the two items and zero production cost. The seller doesn't observe buyer's valuation but knows the marginal distribution of the individual items. The seller doesn't know the correlations between (the joint distribution of) the valuations of the two items. In this situation, we consider seller as a pessimist who maximizes the worst-case expected profit, which means that he will choose a mechanism that maximizes expected profit given that the joint distribution he conjectures will generate a minimum profit under the mechanism he chooses over the entire class of mechanisms.

Following Manelli et al. (2006), , we consider a determinisitic mechanism which is called price schedule. A price schedule is a collection of prices, one price per bundle, which in the two-item cases here can be represented by $(p_1, p_2, p_b) \in R^3_+$, where p_1, p_2 are the price for getting each good in isolation, p_b is the price for getting two items as a bundle.[2] The buyer can choose the option he likes or quit(the outside option) but he can only choose one out of these options.

This mechanism can be served as a self-selection device such that when a buyer choose one out of the three options, it implicitly satisfies two incentive compatibility (IC) constraint and an individual rationality (IR) constraint. For example, if a buyer choose the first option, he gets the first item, pays the price p_1, and his type (v_1, v_2) will satisfy the following constraints:

$$v_1 - p_1 \geq 0 \qquad (IR)$$

$$v_1 - p_1 \geq v_1 + v_2 - p_b \qquad (IC_1)$$

[1] Additive valuation means that the valuation for getting both items are the sum of the valuation for each item.

[2] p_1, p_2, p_3 are all restricted to be finite. Indeed either one of the options can be not provided, which means the posting price for that is just infinite. But for that option we can also provide a very high finite price such that still no buyer will choose it. For example, in the literature, offering only a price p_b for the bundle is called pure bundling. But nothing will change if we offer a price schedule (p_b, p_b, p_b). Thus there is no lost in our definition and we are not going into the concept of mixed bundling or pure bundling.

$$v_1 - p_1 \geq v_2 - p_2 \qquad (IC_2)$$

If a seller offers a price schedule p_1, p_2, p_b such that $p_1 + p_2 = p_b$, it's easy to check this mechanism is equivalent to selling each item independently with p_1 and p_2 respectively. If $p_1 + p_2 > p_b$, the bundled option is offered with a discount. If $p_1 + p_2 < p_b$, the bundled option is offered with a premium. We assumes that the seller can monitor the purchase. This means that p_b can be greater than $p_1 + p_2$. If the purchase can't be monitored and $p_1 + p_2 < p_b$, the buyers who want to buy both items will not choose the bundled option but to buy each item separately.

If a buyer is indifferent between two options, we assume that she will choose the one that generate more profit for the seller. In reality, the seller can achieve this by providing a tiny discount for that profitable option.

3 An example with discrete type buyers

We consider an example that a buyer's valuation distribution for each item are identical as the following:

$$v_1 = \begin{cases} 1 & \textbf{with prob. } a \\ 2 & \textbf{with prob. } 1-a \end{cases}$$

$$v_2 = \begin{cases} 1 & \textbf{with prob. } a \\ 2 & \textbf{with prob. } 1-a \end{cases}$$

The seller knows exactly the distibution for each item, but don't know the correlations between the two items' valuations, which is measured by b in the following table:

		v_2 = 1	v_2 = 2
v_1	1	b	$a-b$
	2	$a-b$	$1-2a+b$

Where $\frac{1}{4} \leq a \leq \frac{3}{4}$ [3] and $\max\{0, 2a-1\} < b < a$. If $b = a^2$, then the distribution of v_1 and v_2 are independent with each other. Also observe that the correlation coefficient of v_1 and v_2 is $\rho(v_1, v_2) = \frac{b-a^2}{a(1-a)}$. If $b = a$, the joint distribution is perfectly positive correlated. If $b = 0$ and $a = \frac{1}{2}$, then the distribution is perfectly negative correlated.

In this example we consider optimal mechanisms within the class of symmetric mechanisms in which the seller will set the same price for individual item, i,e, $p_1^* = p_2^* = p$. These mechanisms are not necessary 'the optimal' but we can still get some insights from them. We consider two situations in which the seller knows the exact joint distribution (knows a and b) and the seller only knows the marginal distribution (knows a).

3.1 The seller knows the joint distribution

In the one-dimensional problem, sell's optimal strategy will be affected by the fraction of low-type consumers. If the fraction of low-type consumers is very low to some extent, the seller can aviod to serve low type consumers and extract all the surplus of high type consumers. This also has some implications in the two items case here. The optimal price schedule by the seller will depend on both a and b as the following:

If $\frac{1}{4} \leq a \leq \frac{1}{2}$, then the optimal mechaism and its expected profit is ($\varepsilon \geq 0$):

$$(p^*, p_b^*, E\pi^*) = \begin{cases} (2+\varepsilon, 3, 3(1-b)) & \text{if } \ b \leq \frac{4a-1}{3} \\ (2, 4, 4-4a) & \text{if } \ \frac{4a-1}{3} < b < \frac{1}{2} \\ (1, 3, 3-4a+2b) & \text{if } \ b \geq \frac{1}{2} \end{cases}$$

[3] Here we restict a to be between $\frac{1}{4}$ and $\frac{3}{4}$ just to let the result to be concise. a can be less than $\frac{1}{4}$ or $\frac{3}{4}$ but the result will be similar and analysis will not be affected.

If $\frac{1}{2} \leq a \leq \frac{3}{4}$, then the optimal mechaism and its expected profit is ($\varepsilon \geq 0$)

$$(p^*, p_b^*, E\pi^*) = \begin{cases} (2+\varepsilon, 3, 3(1-b)) & \text{if } \ b \leq \frac{1}{3} \\ (1+\varepsilon, 2, 2) & \text{if } \ \frac{1}{3} < b < \frac{4a-1}{2} \\ (1, 3, 3-4a+2b) & \text{if } \ b \geq \frac{4a-1}{2} \end{cases}$$

There are totally four price schedules here. We observe that mechanism $(2, 2, 4)$ can be implemented by a separately selling strategy (selling each good separately at price 2). $(2+\varepsilon, 2+\varepsilon, 3)$ can be implemented by a bundling strategy with bundle discount (selling the bundle option at price 3 and the individual good option at price greater than or equal to 2). $(1, 1, 3)$ can be implemented by a bundling strategy with bundle premium(selling each good at price 1 and selling bundle option at price 3). $(1+\varepsilon, 1+\varepsilon, 2)$ can either be implemented by a separately selling strategy (selling each good at price 1 when $\varepsilon = 0$) or a bundling strategy with bundle discount (selling the bundle option at price 2 when $\varepsilon > 0$), but they will generate the same expected profit, thus we can always choose to sell separately to implement $(1+\varepsilon, 1+\varepsilon, 2)$.

3.2 The seller only knows the marginal distribution

If the seller doesn't know the value of b, but only knows a, she can compute that for different joint distribution (determined by b and a) that is consistent with marginal distribuiton (determined by a), the expected profit from any price schedule. For a specific price schedule, a pessimistic seller will want to know the lowest expected profit, i.e. worst-case performance, that can be generated by this mechanism among all the possible joint distribution. The following table gives the expected profit generated by these four mechanisms, for three different values of b when $a \leq \frac{1}{2}$.

$a \leq \frac{1}{2}$	$(2,2,4)$	$(2+\varepsilon, 2+\varepsilon, 3)$	$(1,1,3)$	$(1+\varepsilon, 1+\varepsilon, 2)$
$b = a^2$(Independent)	$4-4a$	$3-3a^2$	$3+2a^2-4a$	2
$b = a$ (Perfectly positive correlated)	$4-4a$	$3-3a$	$3-2a$	2
$b = 0$ (Negative correlated)	$4-4a$	3	$3-4a$	2
Worst-case expected profit	$4-4a$	$3-3a$	$3-4a$	2

It's easy to see that when $a \leq \frac{1}{2}$, if the seller conjectures that b can be these three different values, then the mechanism $(2,2,4)$ will be chosen. If the seller thinks that b actually could be any value that is consistent with the marginal distribution, he will do the procedure for any possible b. It can be verified that the mechanism chosen after considering all possible values of b will be:

$$(p^*, p_b^*, E\pi^*) = \begin{cases} (2, 4, 4-4a) & \text{if } \frac{1}{4} \leq a \leq \frac{1}{2} \\ (1+\varepsilon, 2, 2) & \text{if } \frac{1}{2} < a < \frac{3}{4} \end{cases}$$

We can see that both these mechanisms can be implemented by separately selling strategies. They are robust since each of them will generate the same expected profit for any joint distribution. This is because separately selling doesn't need to care about the joint distribution. The idea that separately selling will always be chosen is actually very simple. The seller can always conjecture that the joint distribution will be positive correlated such that choosing other bundling strategies will not generate more profit than selling separately. It's easy to see that when distributions are perfectly positive correlated, bundling will generate less profit if the bundling price is not the sum of optimal selling price for each individual good. But it's not necessarily sufficient to prove it, since for a given marginal distribution there may not exist a situation that the valuations are perfectly positive correlated.

4 General model with continuous types

In this section, we give a more general model that buyer's type is continous. The seller knows that the marginal distribution of the two items are respectively $F_1(.) \in \Omega_1$ and $F_2(.) \in \Omega_2$. $F_1(.)$ and

$F_2(.)$ are both continuous and have strictly positive density functions $f_1(.)$ and $f_2(.)$, respectively. For any $p \in [0,1]$, $p - \frac{1-F_1(p)}{f_1(p)}$ and $p - \frac{1-F_2(p)}{f_2(p)}$ are both increasing.[4] Both the domain of v_1 and v_2 are normalized to be $[0,1]$. Denote a price schedule for two items as $P = (p_1, p_2, p_b) \in R^3_+$. p_1, p_2 and p_b are all finite. Denote a joint distribution as $G(.) \in \Sigma$ and $g(.)$ as its density function if it exists.

Denote $E\pi(P, G)$ as the expected profit when the joint distribution is $G(.)$ and offers the price schedule P. It is expressed by

$$\begin{aligned} E\pi(P,G) =& p_1 \cdot Pr(v_1 - p_1 \geq 0, v_1 - p_1 \geq v_2 - p_2, v_1 - p_1 \geq v_1 + v_2 - p_b) + \\ & p_2 \cdot Pr(v_2 - p_2 \geq 0, v_2 - p_2 \geq v_1 - p_1, v_2 - p_2 \geq v_1 + v_2 - p_b) + \\ & p_b \cdot Pr(v_1 + v_2 - p_b \geq 0, v_1 + v_2 - p_b \geq v_1 - p_1, v_1 + v_2 - p_b \geq v_2 - p_2) \end{aligned}$$

Denote $E\pi(P, F_1, F_2)$ as the worst-case expected profit when the seller only knows the marginal distribution and offers the price schedule P. Since the seller will consider the worst-case performance when facing the uncertainty, we have

$$E\pi(P, F_1, F_2) = \min_{G \in \Sigma} E\pi(P, G)$$

$$s.t \quad \int_0^1 g(v_1, v_2) dv_2 = f_1(v_1) \quad \int_0^1 g(v_1, v_2) dv_1 = f_2(v_2) \;\; \forall \;\; v_1, v_2 \in [0,1]$$

Then the maximal worst-case expected profit the seller can get is

$$E\pi(P^*, F_1, F_2) = \max_{P \in R^3_+} E\pi(P, F_1, F_2) = \max_{P \in R^3_+} \min_{G \in \Sigma} E\pi(P, G)$$

$$s.t \quad \int_0^1 g(v_1, v_2) dv_2 = f_1(v_1) \quad \int_0^1 g(v_1, v_2) dv_1 = f_2(v_2) \;\; \forall \;\; v_1, v_2 \in [0,1]$$

This maxmin function actually can be interpreted as the payoff function of a zero-sum game between the seller and a nature player. The nature player's strategy is to choose a joint distribution that is consistent with marginal distribution to minimize the seller's expected profit.

[4]These are standard single crossing conditions in the single-dimension problem in order to maintain that the optimal selling price of individual good exists.

First we can furtherly restrict out mechanism space to the following class

$$P_f = \{(p_1, p_2, p_b) \; | 0 \leq p_1 \leq 1, 0 \leq p_2 \leq 1, 0 \leq p_b - p_1 \leq 1, 0 \leq p_b - p_2 \leq 1\}$$

Since the upper bound of each item's valuation is 1, if the price for an individual item option is over 1 or the price for a bundle option is over 2, then this option is never choosen by any consumer and it's equivalent to set the price to be the upper bound. For the same reason, if $p_1 > p_b$ or $p_2 > p_b$, it's equivalent to set $p_1 = p_b$ or $p_2 = p_b$.

If the seller applys a price schedule which can be implemented by the strategy of separately selling, it can be represented by $P \in P_s = \{(p_1, p_2, p_b) | p_1 + p_2 = p_b, 0 \leq p_1 \leq 1, 0 \leq p_2 \leq 1\} \subseteq P_f$. Given the marginal distribution $F_1(.)$ and $F_2(.)$, the expected profit for a mechanism $P \in P_s$ under any possible joint distribution (that is consistent with the marginal distribution) are all the same as the following:

$$E\pi(P, F_1, F_2) = p_1(1 - F_1(p_1)) + p_2(1 - F_2(p_2))$$

This expected profit is maximized when $p_1 = p_1^*$ and $p_2 = p_2^*$, where

$$p_1^* = \arg\max_{p_1 \geq 0} p_1(1 - F_1(p_1))$$

$$p_2^* = \arg\max_{p_2 \geq 0} p_2(1 - F_2(p_2))$$

We can denote $E\pi(P_s^*, F_1, F_2) = p_1^*(1 - F_1(p_1^*)) + p_2^*(1 - F_2(p_2^*))$ as the optimal profit generated by applying a separate selling strategy. To prove that $E\pi(P_s^*, F_1, F_2)$ is the largest expected profit the seller can get when the seller is uncertain about the correlations, the idea is the following. Fix an arbitrary mechanism $P \in P_F$, given the marignal distribution $F_1(.)$ and $F_2(.)$, we find one joint distrbution $G(.)$ which is consistent with the marginal distribution gives an expected profit no more than $E\pi(P_s^*, F_1, F_2)$. This can be represented by

$$E\pi(P, F_1, F_2) = \min_{G \in \Sigma} E\pi(P, G) \leq E\pi(P, G) \leq E\pi(P_s^*, F_1, F_2)$$

Then it must be true that

$$E\pi(P^*, F_1, F_2) = E\pi(P_s^*, F_1, F_2)$$

The proof process is given in the next section.

5 Proof

First, given the marignal distribution $F_1(.)$ and $F_2(.)$, for an arbitrary mechanism $P = (p_1, p_2, p_b) \in P_F$, we define the following sets:[5]

$$A = \{(v_1, v_2)|v_1 > p_1\}$$

$$B = \{(v_1, v_2)|v_2 > p_b - p_1\}$$

$$C = \{(v_1, v_2)|v_1 - v_2 > p_1 - p_2\}$$

$$D = \{(v_1, v_2)|v_2 > p_2\}$$

$$E = \{(v_1, v_2)|v_1 > p_b - p_2\}$$

$$F = \{(v_1, v_2)|v_1 + v_2 > p_b\}$$

A or D are respectively the set of buyer types that are willing to buy item 1 or item 2 without considering other options. B or E are respectively the set of buyer types that are willing to buy item 2 or item 1 if he is already prepared to buy item 1 or item 2. $p_b - p_1$ or $p_b - p_2$ are respectively the implicit price of item 2 or item 1 to a buyer already prepared to buy item 1 or item 2. C is the set of buyer types that is more willing to buy item 1 instead of item 2 without considering the bundled option. F is the set of buyer types that is willing to buy the bundled option without considering other options. Then seller's expected profit can be rewritten as

$$E\pi(P, G) = p_1 \cdot Pr(A \cap B^c \cap C) + p_2 \cdot Pr(D \cap E^c \cap C^c) + p_b \cdot P_r(F \cap B \cap E)$$

[5]In all sets defined, we ignore all tied buyer types, for example $\{(v_1, v_2)|v_1 = p_1\}$, since all sets of tied buyer types in the continuous setting will have measure zero and will not affect the calculation of probability, we ignore them for conciseness in writing.

By the given marginal distribution, we have

$$Pr(A) = 1 - F_1(p_1)$$

$$Pr(B) = 1 - F_2(p_b - p_1)$$

$$Pr(D) = 1 - F_2(p_2)$$

$$Pr(E) = 1 - F_1(p_b - p_2)$$

Consider another selling strategy by the seller: selling the item 1 independently at price p_1 with probability $1-\lambda_1$ and selling the item 1 independently at the implicit price $p_b - p_2$ with probabiliy λ_1; at the same time, selling the item 2 independently at price p_2 with probability $1-\lambda_2$ and selling the item 2 independently at the implicit price $p_b - p_1$ with probability λ_2. This selling strategy will not generate expected profit higher than $E\pi(P_s^*, F_1, F_2)$. And we want to find $0 \le \lambda_1 \le 1$, $0 \le \lambda_2 \le 1$ and a joint distritbuion $G(.)$ such that the expected profit under $G(.)$ and P will not exceed the expected profit of this strategy. This is represented by the following:

$$\begin{aligned} E\pi(P, G) &\le (1-\lambda_1)p_1 \cdot Pr(A) + \lambda_1(p_b - p_2) \cdot Pr(E) + (1-\lambda_2)p_2 \cdot Pr(D) + \lambda_2(p_b - p_1) \cdot Pr(B) \\ &\le \max\{p_1 \cdot Pr(A), (p_b - p_2) \cdot Pr(E)\} + \max\{p_2 \cdot Pr(D), (p_b - p_1) \cdot Pr(B)\} \\ &\le p_1^*(1 - F_1(p_1^*)) + p_2^*(1 - F_2(p_2^*)) \\ &= E\pi(P_s^*, F_1, F_2) \end{aligned} \tag{1}$$

If λ_1, λ_2 are both indeed between 0 and 1, then the second inequality of (1) holds. First we rewrite the RHS of the first inequality of (1):

$$\begin{aligned} &(1-\lambda_1)p_1 \cdot Pr(A) + \lambda_1(p_b - p_2) \cdot Pr(E) + (1-\lambda_2)p_2 \cdot Pr(D) + \lambda_2(p_b - p_1) \cdot Pr(B) \\ =& p_1((1-\lambda_1)Pr(A) - \lambda_2 Pr(B)) + p_2((1-\lambda_2)Pr(D) - \lambda_1 Pr(E)) + p_b(\lambda_1 Pr(E) + \lambda_2 Pr(B)) \end{aligned}$$

Then the first inequality of (1) is equivalent to

$$p_1((1-\lambda_1)Pr(A) - \lambda_2 Pr(B) - Pr(A \cap B^c \cap C)) + p_2((1-\lambda_2)Pr(D) - \lambda_1 Pr(E) - Pr(D \cap E^c \cap C^c))$$
$$+ p_b(\lambda_1 Pr(E) + \lambda_2 Pr(B) - Pr(F \cap B \cap E)) \geq 0$$

Then a group of sufficient conditions for (1) will be

$$(1-\lambda_1)Pr(A) - \lambda_2 Pr(B) \geq Pr(A \cap B^c \cap C)$$

$$(1-\lambda_2)Pr(D) - \lambda_1 Pr(E) \geq Pr(D \cap E^c \cap C^c)$$

$$\lambda_1 Pr(E) + \lambda_2 Pr(B) \geq Pr(F \cap B \cap E)$$

Since $Pr(A \cap B^c \cap C) \leq Pr(A \cap B^c) = Pr(A) - Pr(A \cap B)$ and $Pr(D \cap E^c \cap C^c) \leq Pr(D \cap E^c) = Pr(D) - Pr(D \cap E)$. Then the sufficient conditions become:

$$\lambda_1 Pr(A) + \lambda_2 Pr(B) \leq Pr(A \cap B) \tag{2}$$

$$\lambda_1 Pr(E) + \lambda_2 Pr(D) \leq Pr(D \cap E) \tag{3}$$

$$\lambda_1 Pr(E) + \lambda_2 Pr(B) \geq Pr(F \cap B \cap E) \tag{4}$$

From now on, we prove separately when $p_1 + p_2 < p_b$ (bundling with a premium) and $p_1 + p_2 > p_b$ (bundling with a discount) .

5.1 Bundling with premium ($p_1 + p_2 < p_b$)

If $p_1 + p_2 < p_b$, first we have $Pr(E) \leq Pr(A)$, $Pr(B) \leq Pr(D)$ and $Pr(F \cap B \cap E) = Pr(B \cap E)$. Figure 1 shows that $Pr(A \cap B) = Pr(\text{Rectangle acfd})$, $Pr(D \cap E) = Pr(\text{Rectangle abhg})$, $Pr(F \cap B \cap E) = Pr(B \cap E) = Pr(\text{Rectangle abed})$.

We can assume $Pr(A) \neq 0$,$Pr(B) \neq 0$,$Pr(E) \neq 0$,$Pr(D) \neq 0$,[6] We further assume that

[6] If $Pr(A) = 0$ or $Pr(D) = 0$, $\lambda_1 = 0$ and $\lambda_2 = 0$ will satisfy (2), (3), (4) for any joint distribution ; if $Pr(D) \neq 0$, $Pr(A) \neq 0$ and $Pr(B) = 0$ or $Pr(E) = 0$, then with our assumption a condition that let λ_1 and λ_2 exist is $Pr(B \cap E) \leq Pr(B)Pr(E)$.

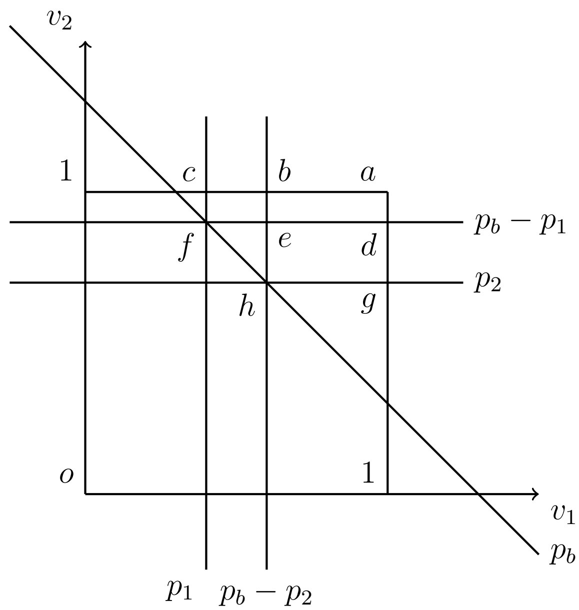

Figure 1

$Pr(A \cap B) = Pr(A)Pr(B)$ and $Pr(D \cap E) = Pr(D)Pr(E)$, then we can plot the constraints (2), (3) in Figure 2.

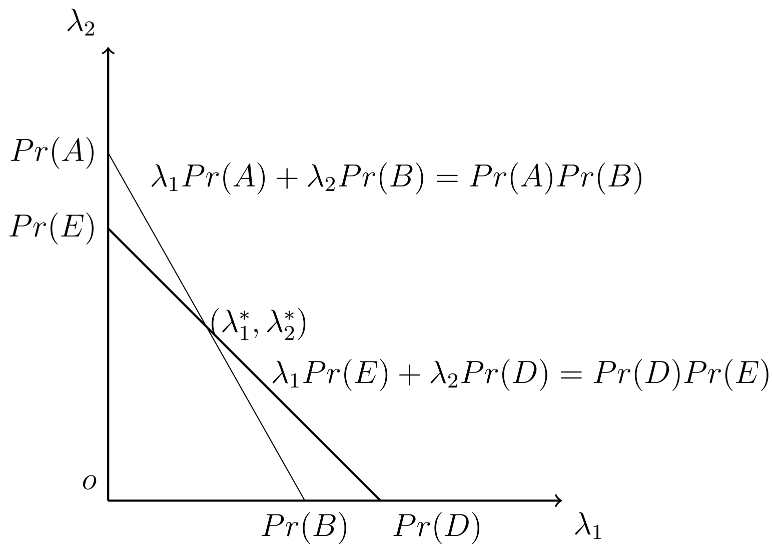

Figure 2

From Figure 2 we can see that if λ_1, λ_2 exist to satisfy (2) (3) (4) then it must be true that $0 \leq \lambda_1 \leq 1$, $0 \leq \lambda_2 \leq 1$. Then the second inequality of (1) holds. What's more, because the relationships between the absolute value of slopes of these three constraints are $\frac{Pr(E)}{Pr(D)} \leq \frac{Pr(E)}{Pr(B)} \leq$

$\frac{Pr(A)}{Pr(B)}$. To make (4) also satisfied we only need to check

$$\lambda_1^* Pr(E) + \lambda_2^* Pr(B) \geq Pr(B \cap E)$$

where $(\lambda_1^*, \lambda_2^*)$ is the intersection point between constraint (2) and (3) and[7]

$$\lambda_1^* = \frac{Pr(D)Pr(B)[Pr(A) - Pr(E)]}{Pr(A)Pr(D) - Pr(B)Pr(E)}$$

$$\lambda_2^* = \frac{Pr(A)Pr(E)[Pr(D) - Pr(B)]}{Pr(A)Pr(D) - Pr(B)Pr(E)}$$

Then we have

$$\frac{Pr(B)Pr(E)\{Pr(D)Pr(A) - Pr(D)Pr(E) + Pr(A)Pr(D) - Pr(B)Pr(A)\}}{Pr(A)Pr(D) - Pr(B)Pr(E)} \geq Pr(B \cap E)$$

It's easy to check that a sufficient condition to hold the inequality above is

$$Pr(B)Pr(E) \geq Pr(B \cap E)$$

If we let $Pr(B \cap E) = Pr(B)Pr(E)$ then the first inequality of (1) will also hold. Combined with $Pr(A \cap B) = Pr(A)Pr(B)$ and $Pr(D \cap E) = Pr(D)Pr(E)$, then it's easy to conjecture that independent distribution will satisfy these conditions. Then we can easily set the joint distribution as independent instributed as the following:

$$g(v_1, v_2) = f_1(v_1)f_2(v_2) \quad \forall \;\; v_1, v_2 \in [0, 1]$$

Thus we prove that for any price schedule that offers a bundling premium there always exists a joint distribution (independent distribution satisfies for all mechanisms) such that the expected profit of it will not exceed the expected profit of optimal separately selling strategy. This supplements McAfee et al. (1989)'s result in the following way. They start from an optimal separately

[7] $Pr(A)Pr(D) - Pr(B)Pr(E) = 0$ happnes only when $Pr(E) = Pr(A)$ and $Pr(B) = Pr(D)$. Then (2) (3) coincide under our assumption and this condition is still sufficient.

selling schedule $(p_1^*, p_2^*, p_1^* + p_2^*)$ and identify sufficient conditions that provide profitable deviation. For one class of sufficient conditions (independence belongs to this class), bundling (with discount) will dominates separately selling. For another class of sufficient conditions (independence doesn't belong to this class), bundling (with premium) will dominate separately selling. We supplement these by illustrating that bundling (with premium) will never dominate separately selling if the product values are independently distributed.

5.2 Bundling with discount ($p_1 + p_2 > p_b$)

If $p_1 + p_2 > p_b$, first we have $Pr(A) \leq Pr(E)$, $Pr(D) \leq Pr(B)$. Like what we did in the last section, we can also assume $Pr(A) \neq 0$, $Pr(B) \neq 0$, $Pr(E) \neq 0$, $Pr(D) \neq 0$. We can plot the constraints (2) and (3) in the figure below.

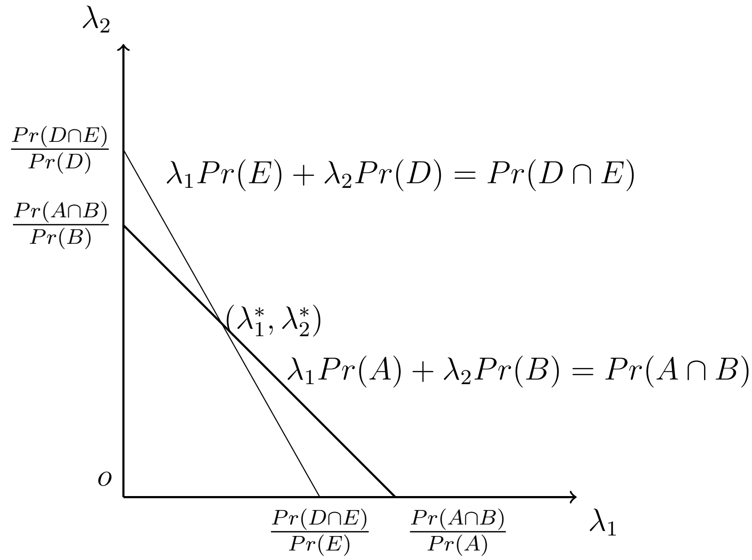

Figure 3

By the similar reason we have that to make (4) also satisfied we only need to check

$$\lambda_1^* Pr(E) + \lambda_2^* Pr(B) \geq Pr(F \cap B \cap E)$$

A more sufficient condition will be

$$\lambda_1^* Pr(E) + \lambda_2^* Pr(B) \geq Pr(B \cap E)$$

This is equivalent to

$$\frac{Pr(B)Pr(D \cap E)[Pr(E) - Pr(A)] + Pr(E)Pr(A \cap B)[Pr(B) - Pr(D)]}{Pr(A)Pr(D) - Pr(B)Pr(E)} \geq Pr(B \cap E)$$

We can denote $Pr(A) = aPr(E) \leq a$, $Pr(D) = bPr(B) \leq b$, where $0 \leq a \leq 1$, $0 \leq b \leq 1$ and $ab < 1$.[8] Then the condition above becomes

$$(1 - a)Pr(D \cap E) + (1 - b)Pr(A \cap B) \geq (1 - ab)Pr(B \cap E)$$

If we let $Pr(D \cap E) = Pr(A \cap B) = Pr(B \cap E)$, since $1 - a + 1 - b \geq 1 - ab$, the condition will be satisfied. Let's see what does this mean in a graph.

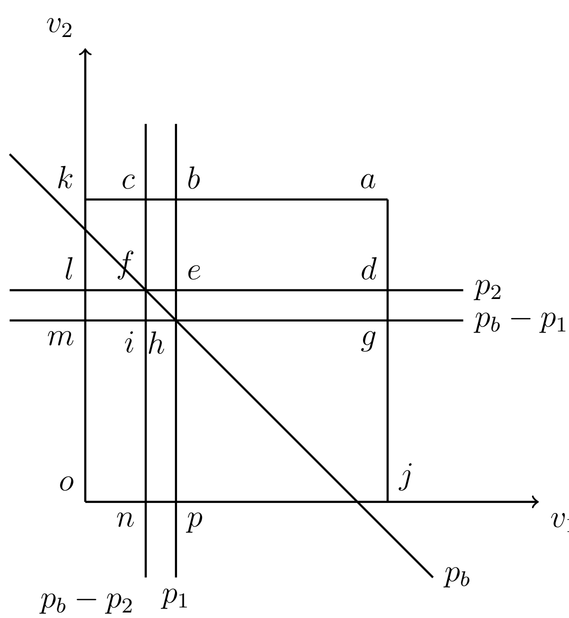

Figure 4

The figure above (for the case $p_b < p_2 + p_1$) shows that $Pr(A \cap B) = Pr(\text{Rectangle abhg})$, $Pr(D \cap E) = Pr(\text{Rectangle acfd})$ and $Pr(B \cap E) = Pr(\text{Rectangle acig})$. Then $Pr(D \cap E) = Pr(A \cap B) = Pr(B \cap E)$ means that $Pr(\text{Rectangle bcfe}) = Pr(\text{Rectangle dehg}) = Pr(\text{Rectangle efih}) = 0$.

[8] We exclude that case that $a = 1$ and $b = 1$, since if this is the case, then $Pr(D) = Pr(B)$, $Pr(A) = Pr(E)$, our condition will be automatically satisfied.

This can be achieved by specifying the density function for each area as follows:

$$g(v_1, v_2) = \begin{cases} f(v_1)f(v_2)\frac{Pr(cklf)}{Pr(D)(1-Pr(E))} \quad \text{if} \quad (v_1, v_2) \in cklf \\ f(v_1)f(v_2)\frac{Pr(abed)}{Pr(D)(1-Pr(A))} \quad \text{if} \quad (v_1, v_2) \in abed \\ f(v_1)f(v_2)\frac{Pr(flmi)}{(Pr(B))-Pr(D))(1-Pr(E))} \quad \text{if} \quad (v_1, v_2) \in flmi \\ f(v_1)f(v_2)\frac{Pr(imon)}{(1-Pr(B))(1-Pr(E))} \quad \text{if} \quad (v_1, v_2) \in imon \\ f(v_1)f(v_2)\frac{Pr(himp)}{(1-Pr(B))(Pr(A)-Pr(E))} \quad \text{if} \quad (v_1, v_2) \in himp \\ f(v_1)f(v_2)\frac{Pr(ghpj)}{(1-Pr(B))(1-Pr(A))} \quad \text{if} \quad (v_1, v_2) \in ghpj \\ 0 \quad \text{if} \quad (v_1, v_2) \in efih \cup bcfe \cup dehg \end{cases}$$

Then we complete the proof that by only knowing about the marginal distribution, the optimal selling strategy when pursuing performance guarantee is to sell separately.

6 Conclusion and Future Work

This paper proposes separately selling as a robust mechanism for a multiproduct seller facing uncertainty of the correlations between product values. In the model, a deterministic mechanism, i.e. price schedule, is offered by a seller to a buyer which has private information about product values. We prove that in the two-items case with continous consumer types, to maximize the worst-case expected profit, the best strategy is to sell independently if the seller only knows the marginal distribution of each item's valuation. The proof consist of two parts. The first part proves that for any mechanism that demands bundle premium, there exists a joint distribution, especially independent distribution applying for all mechanisms, that generates expected profit no more than that of optimally separately selling each item. This shows that if the valuations are independently distributed and known by the seller, offering bundling premium is always dominated by separately selling. The second part proves that for any mechanism that offers bundle discounts, there also exists a joint distribution that generates expected profit no more than that of optimally separately selling each item. One direction of the future work for this project could be extending the two-item case to the case of arbitrary number of items. Another direction could be that assuming the seller has different size of uncertainty on the distribution of buyers' valuations and testing the effect of the uncertainty size on seller's 'optimal' pricing strategy.

References

Adams, W. J. and Yellen, J. L. (1976). Commodity bundling and the burden of monopoly. *The Quarterly Journal of Economics*, 90(3):475–498.

Armstrong, M. (1999). Price discrimination by a many-product firm. *The Review of Economic Studies*, 66(1):151–168.

Armstrong, M. and Vickers, J. (2010). Competitive non-linear pricing and bundling. *The Review of Economic Studies*, 77(1):30–60.

Babaioff, M., Immorlica, N., Lucier, B., and Weinberg, S. M. (2014). A simple and approximately optimal mechanism for an additive buyer. In *2014 IEEE 55th Annual Symposium on Foundations of Computer Science*, pages 21–30. IEEE.

Bergemann, D. and Morris, S. (2005). Robust mechanism design. *Econometrica*, pages 1771–1813.

Hart, S. and Nisan, N. (2017). Approximate revenue maximization with multiple items. *Journal of Economic Theory*, 172:313–347.

Hart, S. and Reny, P. J. (2015). Maximal revenue with multiple goods: Nonmonotonicity and other observations. *Theoretical Economics*, 10(3):893–922.

Kobayashi, B. H. (2005). Does economics provide a reliable guide to regulating commodity bundling by firms? a survey of the economic literature. *Journal of Competition Law & Economics*, 1(4):707–746.

Kolay, S. and Shaffer, G. (2003). Bundling and menus of two-part tariffs. *The Journal of Industrial Economics*, 51(3):383–403.

Manelli, A. M. and Vincent, D. R. (2007). Multidimensional mechanism design: Revenue maximization and the multiple-good monopoly. *Journal of Economic theory*, 137(1):153–185.

Manelli, A. M., Vincent, D. R., et al. (2006). Bundling as an optimal selling mechanism for a multiple-good monopolist. *Journal of Economic Theory*, 127(1):1–35.

McAfee, R. P., McMillan, J., and Whinston, M. D. (1989). Multiproduct monopoly, commodity bundling, and correlation of values. *The Quarterly Journal of Economics*, 104(2):371–383.

Menicucci, D., Hurkens, S., and Jeon, D.-S. (2015). On the optimality of pure bundling for a monopolist. *Journal of Mathematical Economics*, 60:33–42.

Palfrey, T. R. (1983). Bundling decisions by a multiproduct monopolist with incomplete information. *Econometrica: Journal of the Econometric Society*, pages 463–483.

Pavlov, G. (2011). Optimal mechanism for selling two goods. *The BE Journal of Theoretical Economics*, 11(1).